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ANALYSIS OF NONSTATIONARY INCOMPRESSIBLE FLOWS IN DIFFERENTLY SHAPED CHANNELS ON THE BASIS OF SYMMETRIES

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Nonstationary flows in laminar and turbulent regimes in differently shaped channels have been investigated theoretically. An approach has been used which is based on the properties of the symmetry of differential equations (Lie groups) that describe the process of an accelerated channel flow. A way in which the self-similar forms of one-dimensional and two-dimensional flow can be obtained on the basis of symmetries is shown. The self-similar equations of the process and their analytical and numerical solutions are given.

Nonstationary (accelerated) channel flows were usually calculated with the use of operational methods [1], where the velocity distribution was represented in the form of a functional series, but they were not always convenient for calculation. The pressure gradient was assigned in this case by the Heaviside function

$$-\frac{1}{\rho}\frac{dp}{dx}(t)=\begin{cases}0\,, & t\le 0\,,\\ G=\text{const}\,, & t>0\,.\end{cases}$$

But if the pressure gradient is prescribed in the form of an arbitrary function, then frequently the solution cannot be obtained on the basis of operational methods. In [2, 3], group methods of analysis of turbulent flows were used which are based on the application of Lie groups. These groups describe the symmetries of differential equations. We will try to use a similar method to investigate accelerated flows in differently shaped channels provided that $G = G(t)$.

Let us begin with the analysis of a flow in a plane channel provided that $G = G_0t^n$ (G_0 is a constant of corresponding dimension), i.e., the pressure gradient is the exponential function of time. When $n = 0$, the Heaviside function is obtained. The equation of motion has the form

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}=G_0t^n+\nu\frac{\partial^2 u}{\partial y^2}\,. \tag{1}$$

Equation (1) will be investigated on the symmetry in the sense of Lie groups, i.e., let us try to find such transformations that do not alter this equation. Using the procedure described in [4], we find the infinitesimal operator (C_1, C_2, and C_3 are constants)

$$q=(2C_1\,(n+1)\,u+C_2)\,\partial_u+C_1 2t\partial_t+(C_1 y+C_3)\,\partial_y\,, \tag{2}$$

which characterizes the symmetries of Eq. (1) and on the basis of which it is possible to find self-similar variables for this equation. Proceeding from Eq. (2), we compose the differential equation

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$$C_1 2t\frac{\partial\eta}{\partial t}+(C_1 y+C_3)\frac{\partial\eta}{\partial y}=0$$

to find the self-similar coordinate η. The solution of this equation by the method of characteristics yields

$$\eta\sim\frac{y+C_3/C_1}{\sqrt{t}}.$$

In our problems, it is convenient to adopt $C_3 = 0$ and, using the kinematic viscosity, to represent the self-similar coordinate in a dimensionless form:

$$\eta=\frac{y}{\sqrt{\nu t}}. \tag{3}$$

Similarly, we find the self-similar function with account for the fact that in our case it is convenient to select time as a parametric variable. Then, the velocity u can be expressed in terms of the self-similar function, having adopted that $C_2 = 0$:

$$u=\frac{\nu}{h}f(\eta)\frac{t^{n+1}\nu^{n+1}}{h^{2n+2}}. \tag{4}$$

Using (3) and (4), we rearrange Eq. (1) to the ordinary differential equation

$$\frac{d^2f}{d\eta^2}+\frac{\eta}{2}\frac{df}{d\eta}-f(n+1)=-A, \tag{5}$$

in which $A = G_0\nu^{-n-2}h^{2n+3}$. Equation (5) is solved by the method of variation of constants in the form

$$f=D(\eta)_1F_1\left(-1-n,\frac{1}{2};-\frac{\eta^2}{4}\right)+E(\eta)\,\eta_1F_1\left(-\frac{1}{2}-n,\frac{3}{2};-\frac{\eta^2}{4}\right), \tag{6}$$

where ${}_1F_1(a; b; z)$ is the Kummer confluent geometrical function; the functions $D(\eta)$ and $E(\eta)$ are determined by solving the system of equations

$$\frac{dD}{d\eta}\,{}_1F_1\left(-1-n,\frac{1}{2};-\frac{\eta^2}{4}\right)+\frac{dE}{d\eta}E(\eta)\,\eta_1F_1\left(-\frac{1}{2}-n,\frac{3}{2};-\frac{\eta^2}{4}\right)=0,$$

$$\frac{dD}{d\eta}\frac{d_1F_1\left(-1-n,\frac{1}{2};-\frac{\eta^2}{4}\right)}{d\eta}+\frac{dE}{d\eta}\frac{d\eta_1F_1\left(-\frac{1}{2}-n,\frac{3}{2};-\frac{\eta^2}{4}\right)}{d\eta}=-A.$$

Solving this system and substituting the results into Eq. (6), we come to

$$f=A_1F_1\left(-1-n,\frac{1}{2};-\frac{\eta^2}{4}\right)\left(\int_0^\eta\left[6\eta_1F_1\left(-\frac{1}{2}-n,\frac{3}{2};-\frac{\eta^2}{4}\right)d\eta\left\{{}_1F_1\left(-1-n,\frac{1}{2};-\frac{\eta^2}{4}\right)\times\right.\right.\right.$$

$$\times\left[6\,{}_1F_1\left(-\frac{1}{2}-n\,,\frac{3}{2};-\frac{\eta^2}{4}\right)+(1+2n)\,\eta^2\,{}_1F_1\left(\frac{1}{2}-n\,,\frac{5}{2};-\frac{\eta^2}{4}\right)+\right.$$

$$\left.\left.\left.\left.+\,6\,(1+n)\,\eta^2\,{}_1F_1\left(-\frac{1}{2}-n\,,\frac{3}{2};-\frac{\eta^2}{4}\right){}_1F_1\left(-n\,,\frac{3}{2};-\frac{\eta^2}{4}\right)\right]\right\}^{-1}\right]+c_1\right)-$$

$$-A\eta\,{}_1F_1\left(-\frac{1}{2}-n\,,\frac{3}{2};-\frac{\eta^2}{4}\right)\left(\int_0^\eta\left[6\eta\,{}_1F_1\left(-\frac{1}{2}-n\,,\frac{3}{2};-\frac{\eta^2}{4}\right)d\eta\,\left\{{}_1F_1\left(-1-n\,,\frac{1}{2};-\frac{\eta^2}{4}\right)\times\right.\right.\right.$$

$$\times\left[6\,{}_1F_1\left(-\frac{1}{2}-n\,,\frac{3}{2};-\frac{\eta^2}{4}\right)+(1+2n)\,\eta^2\,{}_1F_1\left(\frac{1}{2}-n\,,\frac{5}{2};-\frac{\eta^2}{4}\right)+\right.$$

$$\left.\left.\left.\left.+\,6\,(1+n)\,\eta^2\,{}_1F_1\left(-\frac{1}{2}-n\,,\frac{3}{2};-\frac{\eta^2}{4}\right){}_1F_1\left(-n\,,\frac{3}{2};-\frac{\eta^2}{4}\right)\right]\right\}^{-1}\right]+c_2\right).$$

Using the boundary conditions

$$f=0 \quad \text{for} \quad y=0\,, \quad f=0 \quad \text{for} \quad y=h\,, \tag{7}$$

we find the integration constants c_1 and c_2. In the final form the expression looks like

$$f=A\,{}_1F_1\left(-1-n\,,\frac{1}{2};-\frac{\eta^2}{4}\right)\Phi-A\eta\,{}_1F_1\left(-\frac{1}{2}-n\,,\frac{3}{2};-\frac{\eta^2}{4}\right)\times$$

$$\times\left(\Phi+\frac{\eta_h\,{}_1F_1\left(-\frac{1}{2}-n\,,\frac{3}{2};-\frac{\eta_h^2}{4}\right)-{}_1F_1\left(-1-n\,,\frac{1}{2};-\frac{\eta_h^2}{4}\right)}{\eta_h\,{}_1F_1\left(-\frac{1}{2}-n\,,\frac{3}{2};-\frac{\eta_h^2}{4}\right)}\Phi_h\right), \tag{8}$$

in which

$$\Phi=\int_0^\eta\left[6\eta\,{}_1F_1\left(-\frac{1}{2}-n\,,\frac{3}{2};-\frac{\eta^2}{4}\right)d\eta\,\left\{{}_1F_1\left(-1-n\,,\frac{1}{2};-\frac{\eta^2}{4}\right)\right[6\,{}_1F_1\left(-\frac{1}{2}-n\,,\frac{3}{2};-\frac{\eta^2}{4}\right)+\right.$$

$$\left.\left.\left.+\,(1+2n)\,\eta^2\,{}_1F_1\left(\frac{1}{2}-n\,,\frac{5}{2};-\frac{\eta^2}{4}\right)+6\,(1+n)\,\eta^2\,{}_1F_1\left(-\frac{1}{2}-n\,,\frac{3}{2};-\frac{\eta^2}{4}\right){}_1F_1\left(-n\,,\frac{3}{2};-\frac{\eta^2}{4}\right)\right]\right\}^{-1}\right];$$

$$\Phi_h=\int_0^{\eta_h}\left[6\eta\,{}_1F_1\left(-\frac{1}{2}-n\,,\frac{3}{2};-\frac{\eta^2}{4}\right)d\eta\,\left\{{}_1F_1\left(-1-n\,,\frac{1}{2};-\frac{\eta^2}{4}\right)\right[6\,{}_1F_1\left(-\frac{1}{2}-n\,,\frac{3}{2};-\frac{\eta^2}{4}\right)+\right.$$

$$\left.\left.\left.+\,(1+2n)\,\eta^2\,{}_1F_1\left(\frac{1}{2}-n\,,\frac{5}{2};-\frac{\eta^2}{4}\right)+6\,(1+n)\,\eta^2\,{}_1F_1\left(-\frac{1}{2}-n\,,\frac{3}{2};-\frac{\eta^2}{4}\right){}_1F_1\left(-n\,,\frac{3}{2};-\frac{\eta^2}{4}\right)\right]\right\}^{-1}\right];$$

$$\eta_h = \frac{h}{\sqrt{\nu t}} .$$

The velocity profile can be easily calculated on the basis of Eq. (4) with account for (8) using contemporary applied packages such as "Mathematica." The alternative path is direct numerical integration of Eq. (5) with account for boundary conditions (7), which can be easily implemented using a package of the series "Mathcad." To test the relations obtained, calculations were carried out for $n = 0$. For this variant there is a solution obtained on the basis of the Laplace transformations [1]:

$$u = -\frac{dp/dx}{2\mu}\frac{h^2}{4}\left[\left(1-\left(\frac{y}{h/2}\right)^2\right) - 4\sum_{k=0}^{\infty}(-1)^k \frac{\cos\left[\left(\frac{\pi}{2}+k\pi\right)\frac{y}{h/2}\right]}{\left(\frac{\pi}{2}+k\pi\right)^3}\right]\exp\left[-4\left(\frac{\pi}{2}+k\pi\right)^2 \mathrm{Fo}\right]. \quad (9)$$

In this formula, $\mathrm{Fo} = t\nu/h^2$ is the Fourier number and the coordinate origin in the transverse direction $y = 0$ corresponds to the center of the channel. The comparison of the results by formulas (4) and (9) is given below:

Fo	1/25	1/9	1/4	1	2	5	10	50
Δ, %	0	2.3	9.5	18.6	14.1	6.3	3.2	0.7

Here Δ is the maximum relative difference of the velocities (on the channel axis) calculated by formulas (4) and (9). It is seen that the maximum difference is attained when $\mathrm{Fo} = 1$, and then, as the flow develops, it tends to zero. The advantage of formula (4) is that it allows one to easily calculate the velocity distribution in the accelerated flow in the case of exponential change of the pressure gradient in time.

In the case of a cylindrical channel and exponential variation of the pressure gradient, the equation of motion has the form

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = G_0 t^n + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right). \quad (10)$$

The symmetry of this equation is characterized by the following infinitesimal operator:

$$q = (2C_1(n+1)u + C_2)\,\partial_u + C_1 2t\partial_t + C_1 r\partial_r .$$

The corresponding self-similar variables have the form

$$\eta = \frac{r}{\sqrt{\nu t}}, \quad u = \frac{\nu}{R} f(\eta)\frac{t^{n+1}\nu^{n+1}}{R^{2n+2}},$$

where R is the channel radius. Using these variables, we transform (10) to

$$\frac{d^2 f}{d\eta^2} + \left(\frac{\eta}{2} + \frac{1}{\eta}\right)\frac{df}{d\eta} - f(n+1) = -A .$$

The solution of this equation is sought by the method of variation of constants, using the solution of the corresponding homogeneous equation

$$f = D(\eta)\, {}_1F_1\left(-1-n, 1; -\frac{\eta^2}{4}\right) + E(\eta)\, \mathrm{Gm}\left[\{\{\ \}, \{2+n\}\}, \{\{0, 0\}, \{\ \}\}, \frac{\eta^2}{4}\right], \qquad (11)$$

in which

$$\mathrm{Gm}\left[\{\{a_1, \ldots, a_k\}, \{a_{k+1}, \ldots, a_p\}\}, \{\{b_1, \ldots, b_m\}, \{b_{m+1}, \ldots, b_q\}\}, z\right] = G_{p\,q}^{m\,k}\left(z \left| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \right.\right) =$$

$$= \frac{1}{2\pi i} \oint \frac{\Gamma(1-a_1-s) \ldots \Gamma(1-a_k-s)\, \Gamma(b_1+s) \ldots \Gamma(b_m+s)}{\Gamma(a_{k+1}+s) \ldots \Gamma(a_p+s)\, \Gamma(1-b_{m+1}-s) \ldots (\Gamma(1-b_q-s)} z^{-s}\, ds$$

is the so-called Meijer G function. The integration contour lies between the poles of the functions $\Gamma(1 - a_i - s)$ and $\Gamma(b_i + s)$. The gap in the curly brackets in Eq. (11) means that these coefficients are absent, i.e.,

$$\mathrm{Gm}\left[\{\{\ \}, \{2+n\}\}, \{\{0, 0\}, \{\ \}\}, \frac{\eta^2}{4}\right] = \frac{1}{2\pi i} \oint \frac{\Gamma(s)}{\Gamma(2+n+s)} \left(\frac{\eta^2}{4}\right)^{-s} ds\,.$$

The functions $D(\eta)$ and $E(\eta)$ are determined, just as in the case of a plane channel, from the solution of the corresponding system of equations:

$$\frac{dD}{d\eta}\, {}_1F_1\left(-1-n, 1; -\frac{\eta^2}{4}\right) + \frac{dE}{d\eta}\, \mathrm{Gm}\left[\{\{\ \}, \{2+n\}\}, \{\{0, 0\}, \{\ \}\}, \frac{\eta^2}{4}\right] = 0\,,$$

$$\frac{dD}{d\eta} \frac{n+1}{2} \eta\, {}_1F_1\left(-n, 2; -\frac{\eta^2}{4}\right) - \frac{dE}{d\eta} \frac{\eta}{2}\, \mathrm{Gm}\left[\{\{\ \}, \{1+n\}\}, \{\{-1, 0\}, \{\ \}\}, \frac{\eta^2}{4}\right] = -A\,.$$

Having solved this system and substituted the results into Eq. (11), we obtain an expression for the velocity distribution. For a cylindrical channel the integration constants are found from the boundary conditions

$$f' = 0 \quad \text{for} \quad \eta = 0\,, \quad f = 0 \quad \text{for} \quad \eta = \eta_R = \frac{R}{\sqrt{\nu t}}\,.$$

For an annular channel with internal radius R_1 and external radius R_2, the boundary conditions have the form

$$f = 0 \quad \text{for} \quad \eta = \eta_1 = \frac{R_1}{\sqrt{\nu t}}\,, \quad f = 0 \quad \text{for} \quad \eta = \eta_2 = \frac{R_2}{\sqrt{\nu t}}\,. \qquad (12)$$

The calculations carried out for the cylindrical channel with $n = 0$ show that the results closely coincide with those found on the basis of the relation

$$u = -\frac{dp/dx}{4\mu} R^2 \left[\left(1 - \left(\frac{r}{R}\right)^2\right) - 8 \sum_{k=0}^{\infty} (-1)^k \frac{J_0\left(\alpha_k \frac{r}{R}\right)}{\alpha_k^3 J_1(\alpha_k)}\right] \exp\left[-\alpha_k^2\, \mathrm{Fo}\right],$$

which is obtained by the operational method [1]. Here J_0 and J_1 are the Bessel functions of the first kind of zero and first order, respectively; α_k are the zeros of the function J_0.

Let us consider a curvilinear channel. The equation of motion with an exponential change in the pressure gradient has the following form:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho r}\frac{\partial p}{\partial \varphi} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) = \frac{G_0 t^n}{r} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right). \tag{13}$$

The flow is along the azimuthal coordinate φ. The azimuthal pressure gradient is independent of the radial coordinate and is an exponential function of time. The symmetries of Eq. (13) are characterized by the infinitesimal operator

$$q = (2n+1)\, u\partial_u + 2t\partial_t + r\partial_r\,,$$

which gives the following self-similar variables:

$$\eta = \frac{r}{\sqrt{\nu t}}\,, \quad u = \frac{\nu}{R_1} f(\eta)\,\frac{t^{\frac{2n+1}{2}}\,\nu^{\frac{2n+1}{2}}}{R_1^{2n+1}}\,.$$

Using them, we transform Eq. (13) to

$$\frac{d^2 f}{d\eta^2} + \left(\frac{\eta}{2} + \frac{1}{\eta}\right)\frac{df}{d\eta} - f\left(n + \frac{1}{2} + \frac{1}{\eta^2}\right) = -\frac{B}{\eta}\,, \tag{14}$$

where $B = G_0 \nu^{-n-2} h^{2n+2}$. The solution of Eq. (14) with boundary conditions (12) has the form

$$f = B\,{}_1F_1\left(-n, 2; -\frac{\eta^2}{4}\right)\left(\int_{\eta_1}^{\eta}\left[\mathrm{Gm}\left[\left\{\{\,\}, \left\{\frac{3}{2}+n\right\}\right\}, \left\{\left\{-\frac{1}{2}, \frac{1}{2}\right\}, \{\,\}\right\}, \frac{\eta^2}{4}\right] d\eta \times\right.\right.$$

$$\times\left\{\eta\left[\frac{\eta}{2}\,{}_1F_1\left(-n, 2; -\frac{\eta^2}{4}\right)\mathrm{Gm}\left[\left\{\{-1\}, \left\{\frac{1}{2}+n\right\}\right\}, \left\{\left\{-\frac{3}{2}, -\frac{1}{2}\right\}, \{0\}\right\}, \frac{\eta^2}{4}\right] -\right.\right.$$

$$\left.\left.\left.-\frac{n\eta}{4}\,{}_1F_1\left(1-n, 3; -\frac{\eta^2}{4}\right)\mathrm{Gm}\left[\left\{\{\,\}, \left\{\frac{3}{2}+n\right\}\right\}, \left\{\left\{-\frac{1}{2}, \frac{1}{2}\right\}, \{\,\}\right\}, \frac{\eta^2}{4}\right]\right]\right\}^{-1}\right] + c_1\right) -$$

$$-B\,\mathrm{Gm}\left[\left\{\{\,\}, \left\{\frac{3}{2}+n\right\}\right\}, \left\{\left\{-\frac{1}{2}, \frac{1}{2}\right\}, \{\,\}\right\}, \frac{\eta^2}{4}\right]\left(\int_{\eta_1}^{\eta}\left[{}_1F_1\left(-n, 2; -\frac{\eta^2}{4}\right) d\eta \times\right.\right.$$

$$\times\left\{\eta\left[\frac{\eta}{2}\,{}_1F_1\left(-n, 2; -\frac{\eta^2}{4}\right)\mathrm{Gm}\left[\left\{\{-1\}, \left\{\frac{1}{2}+n\right\}\right\}, \left\{\left\{-\frac{3}{2}, -\frac{1}{2}\right\}, \{0\}\right\}, \frac{\eta^2}{4}\right] -\right.\right.$$

$$\left.\left.\left.-\frac{n\eta}{4}\,{}_1F_1\left(1-n, 3; -\frac{\eta^2}{4}\right)\mathrm{Gm}\left[\left\{\{\,\}, \left\{\frac{3}{2}+n\right\}\right\}, \left\{\left\{-\frac{1}{2}, \frac{1}{2}\right\}, \{\,\}\right\}, \frac{\eta^2}{4}\right]\right]\right\}^{-1}\right] + c_2\right),$$

where

$$c_1 = \mathrm{Gm}\left[\left\{\{\,\}, \left\{\frac{3}{2}+n\right\}\right\}, \left\{\left\{-\frac{1}{2}, \frac{1}{2}\right\}, \{\,\}\right\}, \frac{\eta_1^2}{4}\right]\left\{{}_1F_1\left(-n, 2; -\frac{\eta_2^2}{4}\right)N_1 +\right.$$

$$+\,\mathrm{Gm}\left[\left\{\{\ \},\left\{\frac{3}{2}+n\right\}\right\},\left\{\left\{-\frac{1}{2},\frac{1}{2}\right\},\{\ \}\right\},\frac{\eta_2^2}{4}\right]N_2\Big\}\Big\{{}_1F_1\left(-n\,,2\,;-\frac{\eta_1^2}{4}\right)\times$$

$$\times\,\mathrm{Gm}\left[\left\{\{\ \},\left\{\frac{3}{2}+n\right\}\right\},\left\{\left\{-\frac{1}{2},\frac{1}{2}\right\},\{\ \}\right\},\frac{\eta_2^2}{4}\right]-{}_1F_1\left(-n\,,2\,;-\frac{\eta_2^2}{4}\right)\times$$

$$\times\,\mathrm{Gm}\left[\left\{\{\ \},\left\{\frac{3}{2}+n\right\}\right\},\left\{\left\{-\frac{1}{2},\frac{1}{2}\right\},\{\ \}\right\},\frac{\eta_1^2}{4}\right]\Big\}^{-1}\;;$$

$$c_2=-\left[{}_1F_1\left(-n\,,2\,;-\frac{\eta_1^2}{4}\right)\right]\Big\{{}_1F_1\left(-n\,,2\,;-\frac{\eta_2^2}{4}\right)N_1+$$

$$+\mathrm{Gm}\left[\left\{\{\ \},\left\{\frac{3}{2}+n\right\}\right\},\left\{\left\{-\frac{1}{2},\frac{1}{2}\right\},\{\ \}\right\},\frac{\eta_2^2}{4}\right]N_2\Big]\Big]\Big\{{}_1F_1\left(-n\,,2\,;-\frac{\eta_1^2}{4}\right)\times$$

$$\times\,\mathrm{Gm}\left[\left\{\{\ \},\left\{\frac{3}{2}+n\right\}\right\},\left\{\left\{-\frac{1}{2},\frac{1}{2}\right\},\{\ \}\right\},\frac{\eta_2^2}{4}\right]-{}_1F_1\left(-n\,,2\,,-\frac{\eta_2^2}{4}\right)\times$$

$$\times\,\mathrm{Gm}\left[\left\{\{\ \},\left\{\frac{3}{2}+n\right\}\right\},\left\{\left\{-\frac{1}{2},\frac{1}{2}\right\},\{\ \}\right\},\frac{\eta_1^2}{4}\right]\ \Big\}^{-1}\;;$$

$$N_1=\int_{\eta_1}^{\eta_2}\Big[\mathrm{Gm}\left[\left\{\{\ \},\left\{\frac{3}{2}+n\right\}\right\},\left\{\left\{-\frac{1}{2},\frac{1}{2}\right\},\{\ \}\right\},\frac{\eta^2}{4}\right]d\eta\,\Big\{\eta\Big[\frac{\eta}{2}\,{}_1F_1\left(-n\,,2\,;-\frac{\eta^2}{4}\right)\times$$

$$\times\,\mathrm{Gm}\left[\left\{\{-1\},\left\{\frac{1}{2}+n\right\}\right\},\left\{\left\{-\frac{3}{2},-\frac{1}{2}\right\},\{0\}\right\},\frac{\eta^2}{4}\right]-\frac{n\eta}{4}\,{}_1F_1\left(1-n\,,3\,;-\frac{\eta^2}{4}\right)\times$$

$$\times\,\mathrm{Gm}\left[\left\{\{\ \},\left\{\frac{3}{2}+n\right\}\right\},\left\{\left\{-\frac{1}{2},\frac{1}{2}\right\},\{\ \}\right\},\frac{\eta^2}{4}\right]\ \Big]\Big\}^{-1}\Big];$$

$$N_2=\int_{\eta_1}^{\eta_2}\Big[{}_1F_1\left(-n\,,2\,;-\frac{\eta^2}{4}\right)d\eta\,\Big\{\eta\Big[\frac{\eta}{2}\,{}_1F_1\left(-n\,,2\,;-\frac{\eta^2}{4}\right)\times$$

$$\times\,\mathrm{Gm}\left[\left\{\{-1\},\left\{\frac{1}{2}+n\right\}\right\},\left\{\left\{-\frac{3}{2},-\frac{1}{2}\right\},\{0\}\right\},\frac{\eta^2}{4}\right]-\frac{n\eta}{4}\,{}_1F_1\left(-1-n\,,3\,;-\frac{\eta^2}{4}\right)\times$$

$$\times\,\mathrm{Gm}\left[\left\{\{\ \},\left\{\frac{3}{2}+n\right\}\right\},\left\{\left\{-\frac{1}{2},\frac{1}{2}\right\},\{\ \}\right\},\frac{\eta^2}{4}\right]\ \Big]\Big\}^{-1}\Big].$$

The results of calculations for the case $R_2/R_1 = 1.1$ for $n = 0$ are shown in Fig. 1 in a normalized form. The ordinate axis has the function $f(\eta)$ related to its maximum value. It is seen from Fig. 1 that with the development of the flow (with increase in the Fourier number $\mathrm{Fo} = t\nu/R_1^2$) the maximum of the velocity profile moves from the convex wall to the center of the channel. In the initial stage of development (Fo = 0.01), the velocity profile has a maximum near the convex wall. Thereafter, when the flow has already acquired some

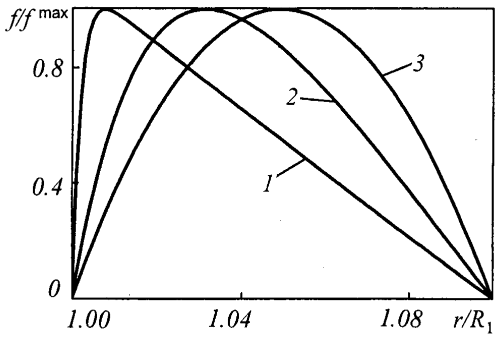

Fig. 1. Velocity profiles of an accelerated flow in a curvilinear channel: 1) Fo = 0.01; 2) 0.1; 3) 1.

initial momentum, this factor (the radial pressure gradient) exerts a not so substantial influence on the shape of the velocity profile, and its maximum tends to the channel center. At large values of the Fourier number (Fo > 1) the nonstationary velocity profile agrees well with the theoretical stationary one [5].

Now, we consider a hypothetical turbulent accelerated flow in a plane channel assuming that the flow induced by a nonstationary pressure gradient is instantly turbulized. Of course, under real conditions the appearing flow always has a laminar character; however, under large pressure gradients the time between the occurrence of the flow and its transition to a turbulent regime can be insignificant. In this case it is possible to neglect this delay in time and consider the flow to be turbulent from the start. The equation of motion has the form

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{\partial}{\partial y}\left[\nu_\Sigma \frac{\partial u}{\partial y}\right] = G_0 t^n + \frac{\partial}{\partial y}\left[\nu_\Sigma \frac{\partial u}{\partial y}\right]. \tag{15}$$

It is assumed here that the pressure gradient has an exponential character, whereas for the total ν_Σ (turbulent and molecular) viscosity we use the model of the mixing length for

$$\nu_\Sigma = \nu + \chi^2 y^2 \frac{\partial u}{\partial y},$$

where χ = 0.4 is the von Kármán constant. The calculation is to be carried out up to the middle of the channel, assuming that the flow is symmetrical relative to the center of the channel.

The symmetries of Eq. (15) for $n = -1.5$ can be described by the infinitesimal operator

$$q = (-C_1 u + C_2)\,\partial_u + C_1 2t\partial_t + C_1 y \partial_y,$$

which yields the following self-similar variables:

$$\eta = \frac{y}{\sqrt{\nu t}}, \quad u = \sqrt{\frac{\nu}{t}}\, f(\eta). \tag{16}$$

Using them, we make Eq. (15) dimensionless:

$$\frac{d}{d\eta}\left[\left(1 + \chi^2\eta^2 \frac{df}{d\eta}\right)\frac{df}{d\eta}\right]\frac{\eta}{2}\frac{df}{d\eta} + \frac{1}{2}f = -\frac{G_0 t^{3/2+n}}{\sqrt{\nu}}. \tag{17}$$

We can see that the total self-similarity exists only when $n = -1.5$, as was noted above. However, this is not an obstacle for numerical integration of (17), since the time can be interpreted as the parameter in integration over η, i.e., to use as if the "marching" method of calculation in time. The procedure of numerical integration of Eq. (7) is easily realized with the aid of the "Mathcad" package under the boundary conditions

$$f = 0 \quad \text{for} \quad y = 0\,, \quad f' = 0 \quad \text{for} \quad y = h/2\,,$$

which reflect the symmetry of the flow relative to the center of the channel.

Equation (17) was obtained assuming the presence of the exponential pressure gradient. However, as we see, this limitation yields the total self-similarity only on the condition that $n = -1.5$. Moreover, neither the variables in (17), nor the left-hand side of (17) depend on n. Consequently, there is no need to restrict the form of the function of the pressure gradient. We may analyze the case where the pressure gradient is an arbitrary function of time:

$$-\frac{1}{\rho}\frac{dp}{dx}(t) = G(t)\,. \tag{18}$$

Using Eqs. (16) and (18), from Eq. (15) we obtain

$$\frac{d}{d\eta}\left[\left(1 + \chi^2\eta^2\frac{df}{d\eta}\right)\frac{df}{d\eta}\right] + \frac{\eta}{2}\frac{df}{d\eta} + \frac{1}{2}f = -\frac{G(t)\,t^{3/2}}{\sqrt{\nu}}\,.$$

This equation is integrated in the same way as Eq. (17), i.e., the term $G(t)t^{3/2}$ is prescribed numerically at each step in time.

We consider the ν-model. In addition to Eq. (15) it also contains an equation that describes the behavior of the total ν_Σ viscosity [6]:

$$\frac{\partial\nu_\Sigma}{\partial t} = \nu_\Sigma\frac{\partial^2\nu_\Sigma}{\partial y^2} + \left(\frac{\partial\nu_\Sigma}{\partial y}\right)^2 + A_k(\nu_\Sigma - j\nu)\left|\frac{\partial u}{\partial y}\right| - \frac{B_k}{L_k^2}(\nu_\Sigma - j\nu)\,\nu_\Sigma\,, \tag{19}$$

where $A_k = 0.133$, $B_k = 0.8$, and $L_k = y$; j may be equal to unity if, in the last two terms in Eq. (19), there is turbulent viscosity, or to zero if the viscosity is total. The group analysis of the system of equations (15) and (19) shows that its symmetries at $j = 0$ are described by the following infinitesimal operator:

$$q = (1 + n)\,u\partial_u + 2t\partial_t + (2 + n)\,y\partial_y + (3 + 2n)\,\nu_\Sigma\,\partial_{\nu_\Sigma}\,.$$

When $j = 1$, we failed to obtain expressions for the groups of symmetry. However, as we can see below, this is not a substantial problem in numerical calculations.

Using the infinitesimal operator, we find the self-similar variables

$$\eta = \frac{yh^{3+2n}}{(\nu t)^{2+n}}\,, \quad u = \frac{\nu^{2+n}\,t^{1+n}}{h^{3+2n}}f(\eta)\,, \quad \nu_\Sigma = \frac{\nu^{4+2n}\,t^{3+2n}}{h^{6+4n}}N(\eta)\,,$$

with the aid of which we transform the system (15), (19) to the self-similar form

$$\frac{d^2f}{d\eta^2}N + \frac{dN}{d\eta}\frac{df}{d\eta} + (2 + n)\,\eta\frac{df}{d\eta} - (n + 1)\,f = -\frac{G_0h^{3+2n}}{\nu^{2+n}}\,,$$

$$\frac{d^2N}{d\eta^2}N + \left(\frac{dN}{d\eta}\right)^2 + (2 + n)\,\eta\frac{dN}{d\eta} + A_k\left(N - \frac{j}{\mathrm{Fo}^{3+2n}}\right)\frac{df}{d\eta} + \frac{B_k}{\eta^2}\left(N - \frac{j}{\mathrm{Fo}^{3+2n}}\right)N - (3 + 2n)\,N = 0\,.$$

As we see, the given system of equations is not completely self-similar because of the presence of the constant j. When $j = 0$, we come to a completely self-similar system of equations. The presence of the non-self-similarity must not cause difficulties in numerical calculation, since at each step of the marching variable it is necessary to prescribe Fo as a constant parameter and perform the solution of the system as of the system of ordinary differential equations.

We consider the k–ε model of turbulent viscosity

$$\nu_\Sigma = \nu + C_\nu \frac{k^2}{\varepsilon} .$$

According to this model, the equation of motion is supplemented with two equations for the kinetic energy of turbulence k and dissipation rate ε [7]:

$$\frac{\partial k}{\partial t} = \frac{\partial}{\partial y}\left[\left(\nu + C_\nu \frac{k^2}{\varepsilon}\right)\frac{\partial u}{\partial y}\right] + C_\nu \frac{k^2}{\varepsilon}\left(\frac{\partial u}{\partial y}\right)^2 - \varepsilon ,$$

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial}{\partial y}\left[\left(\nu + C_\nu \frac{k^2}{\varepsilon}\right)\frac{\partial \varepsilon}{\partial y}\right] + C_\nu C_{1\varepsilon}\, k\left(\frac{\partial u}{\partial y}\right)^2 - C_{2\varepsilon}\frac{\varepsilon^2}{k} , \tag{20}$$

where C_ν, $C_{1\varepsilon}$, and $C_{2\varepsilon}$ are the constants of the model.

When $n = -1.5$, the system of equations (15) and (20) has the symmetries

$$q = \left(\frac{1}{2} C_1 y + C_2\right)\partial_y + C_1 t \partial_t - \left(\frac{1}{2} C_1 u + C_3\right)\partial_u - C_1 k \partial_k - 2C_1 \varepsilon \partial_\varepsilon ,$$

to which there correspond the following self-similar variables:

$$\eta = \frac{y}{\sqrt{\nu t}} ,\quad u = \sqrt{\frac{\nu}{t}}\, f(\eta) ,\quad k = \frac{\nu}{t} K(\eta) ,\quad \varepsilon = \frac{\nu}{t^2} E(\eta) .$$

On their basis we find

$$\left(1 + C_\nu \frac{K^2}{E}\right)\frac{d^2 f}{d\eta^2} + \left(\frac{\eta}{2} - C_\nu \frac{K^2}{E^2}\frac{dE}{d\eta} + 2C_\nu \frac{K}{E}\frac{dK}{d\eta}\right)\frac{df}{d\eta} + \frac{1}{2} f = -\frac{G_0 t^{3/2+n}}{\sqrt{\nu}} ,$$

$$\left(1 + C_\nu \frac{K^2}{E}\right)\frac{d^2 K}{d\eta^2} + \left(\frac{\eta}{2} - C_\nu \frac{K^2}{E^2}\frac{dE}{d\eta} + 2C_\nu \frac{K}{E}\frac{dK}{d\eta}\right)\frac{dK}{d\eta} + \left[1 + C_\nu \frac{K}{E}\left(\frac{df}{d\eta}\right)^2\right] K - E = 0 ,$$

$$\left(1 + C_\nu \frac{K^2}{E}\right)\frac{d^2 E}{d\eta^2} + \left(\frac{\eta}{2} - C_\nu \frac{K^2}{E^2}\frac{dE}{d\eta} + 2C_\nu \frac{K}{E}\frac{dK}{d\eta}\right)\frac{dE}{d\eta} + \left(2 - C_{2\varepsilon}\frac{E}{K}\right) E + C_{1\varepsilon} C_\nu K\left(\frac{df}{d\eta}\right)^2 = 0$$

Just as in the case of the mixing-path model, this model is strictly approximated for $n = -1.5$. Again, this is not an obstacle for numerical integration, since the time can be interpreted as the parameter in integration over η. If Eq. (18) is used for the pressure gradient, we have the equation of motion in a more general form:

$$\left(1 + C_\nu \frac{K^2}{E}\right)\frac{d^2 f}{d\eta^2} + \left(\frac{\eta}{2} - C_\nu \frac{K^2}{E^2}\frac{dE}{d\eta} + 2C_\nu \frac{K}{E}\frac{dK}{d\eta}\right)\frac{df}{d\eta} + \frac{1}{2} f = -\frac{G t^{3/2}}{\sqrt{\nu}} .$$

For a cylindrical channel the accelerated flow is described by the system

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{1}{r}\frac{\partial}{\partial r}\left[\left(\nu + C_\nu \frac{k^2}{\varepsilon}\right) r \frac{\partial u}{\partial r}\right] = G_0 t^n + \frac{1}{r}\frac{\partial}{\partial r}\left[\left(\nu + C_\nu \frac{k^2}{\varepsilon}\right) r \frac{\partial u}{\partial r}\right],$$

$$\frac{\partial k}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[\left(\nu + C_\nu \frac{k^2}{\varepsilon}\right) r \frac{\partial k}{\partial r}\right] + C_\nu \frac{k^2}{\varepsilon}\left(\frac{\partial u}{\partial r}\right)^2 - \varepsilon\,, \quad \frac{\partial \varepsilon}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[\left(\nu + C_\nu \frac{k^2}{\varepsilon}\right) r \frac{\partial \varepsilon}{\partial r}\right] + C_\nu C_{1\varepsilon} k \left(\frac{\partial u}{\partial r}\right)^2 - C_{2\varepsilon}\frac{\varepsilon^2}{k}\,,$$

which has the infinitesimal operator for $n = -1.5$:

$$q = \frac{1}{2} C_1 r \partial_r + C_1 t \partial_t - \left(\frac{1}{2} C_1 u + C_2\right)\partial_u - C_1 k \partial_k - 2C_1 \varepsilon \partial_\varepsilon\,.$$

This operator gives the following self-similar forms:

$$\eta = \frac{r}{\sqrt{\nu t}}\,, \quad u = \sqrt{\frac{\nu}{t}}\, f(\eta)\,, \quad k = \frac{\nu}{t} K(\eta)\,, \quad \varepsilon = \frac{\nu}{t^2} E(\eta)\,,$$

$$\left(1 + C_\nu \frac{K^2}{E}\right)\frac{d^2 f}{d\eta^2} + \left(\frac{\eta}{2} + \frac{1}{\eta} - C_\nu \frac{K^2}{E^2}\frac{dE}{d\eta} + 2C_\nu \frac{K}{E}\frac{dK}{d\eta} + C_\nu \frac{K^2}{E}\frac{1}{\eta}\right)\frac{df}{d\eta} + \frac{1}{2} f = -\frac{G_0 t^{3/2+n}}{\sqrt{\nu}}\,,$$

$$\left(1 + C_\nu \frac{K^2}{E}\right)\frac{d^2 K}{d\eta^2} + \left(\frac{\eta}{2} + \frac{1}{\eta} - C_\nu \frac{K^2}{E^2}\frac{dE}{d\eta} + 2C_\nu \frac{K}{E}\frac{dK}{d\eta} + C_\nu \frac{K^2}{E}\frac{1}{\eta}\right)\frac{dK}{d\eta} + \left[1 + C_\nu \frac{K}{E}\left(\frac{df}{d\eta}\right)^2\right] K - E = 0\,,$$

$$\left(1 + C_\nu \frac{K^2}{E}\right)\frac{d^2 E}{d\eta^2} + \left(\frac{\eta}{2} + \frac{1}{\eta} - C_\nu \frac{K^2}{E^2}\frac{dE}{d\eta} + 2C_\nu \frac{K}{E}\frac{dK}{d\eta} + C_\nu \frac{K^2}{E}\frac{1}{\eta}\right)\frac{dE}{d\eta} + \left(2 - C_{2\varepsilon}\frac{E}{K}\right) E +$$

$$+ C_{1\varepsilon} C_\nu K \left(\frac{df}{d\eta}\right)^2 = 0\,.$$

With the arbitrary form of the pressure gradient function (18), the equation of motion is transformed to

$$\left(1 + C_\nu \frac{K^2}{E}\right)\frac{d^2 f}{d\eta^2} + \left(\frac{\eta}{2} + \frac{1}{\eta} - C_\nu \frac{K^2}{E^2}\frac{dE}{d\eta} + 2C_\nu \frac{K}{E}\frac{dK}{d\eta} + C_\nu \frac{K^2}{E}\frac{1}{\eta}\right)\frac{df}{d\eta} + \frac{1}{2} f = -\frac{G t^{3/2}}{\sqrt{\nu}}\,.$$

In the case of a curvilinear channel the k–ε model has the form

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{1}{r}\frac{\partial}{\partial r}\left[\left(\nu + C_\nu \frac{k^2}{\varepsilon}\right)\left(r\frac{\partial u}{\partial r} - u\right)\right] + \left(\nu + C_\nu \frac{k^2}{\varepsilon}\right)\frac{\partial}{\partial r}\left[\frac{u}{r}\right] =$$

$$= G_0 t^n + \frac{1}{r}\frac{\partial}{\partial r}\left[\left(\nu + C_\nu \frac{k^2}{\varepsilon}\right)\left(r\frac{\partial u}{\partial r} - u\right)\right] + \left(\nu + C_\nu \frac{k^2}{\varepsilon}\right)\frac{\partial}{\partial r}\left[\frac{u}{r}\right],$$

$$\frac{\partial k}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[\left(\nu + C_\nu \frac{k^2}{\varepsilon}\right) r \frac{\partial k}{\partial r}\right] + C_\nu \frac{k^2}{\varepsilon}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right)^2 - \varepsilon\,,$$

$$\frac{\partial\varepsilon}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left[\left(\nu+C_\nu\frac{k^2}{\varepsilon}\right)r\frac{\partial\varepsilon}{\partial r}\right]+C_\nu C_{1\varepsilon}k\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)^2-C_{2\varepsilon}\frac{\varepsilon^2}{k}\,.$$

The symmetries of this system for $n = -1.5$ are

$$q=\frac{1}{2}r\partial_r+t\partial_t-\frac{1}{2}u\partial_u-k\partial_k-2\varepsilon\partial_\varepsilon\,,$$

which makes it possible to obtain

$$\eta=\frac{r}{\sqrt{\nu t}}\,,\quad u=\sqrt{\frac{\nu}{t}}\,f(\eta)\,,\quad k=\frac{\nu}{t}K(\eta)\,,\quad \varepsilon=\frac{\nu}{t^2}E(\eta)\,,$$

$$\left(1+C_\nu\frac{K^2}{E}\right)\frac{d^2f}{d\eta^2}+\left(\frac{\eta}{2}+\frac{1}{\eta}-C_\nu\frac{K^2}{E^2}\frac{dE}{d\eta}+2C_\nu\frac{K}{E}\frac{dK}{d\eta}+C_\nu\frac{K^2}{E}\frac{1}{\eta}\right)\frac{df}{d\eta}+$$

$$+\left(\frac{1}{2}-\frac{1}{\eta^2}-C_\nu\frac{K^2}{E^2}\frac{1}{\eta^2}+C_\nu\frac{K^2}{\eta E^2}\frac{dE}{d\eta}-2C_\nu\frac{K}{\eta E}\frac{dK}{d\eta}\right)f=-\frac{G_0t^{3/2+n}}{\sqrt{\nu}}\,,$$

$$\left(1+C_\nu\frac{K^2}{E}\right)\frac{d^2K}{d\eta^2}+\left(\frac{\eta}{2}+\frac{1}{\eta}-C_\nu\frac{K^2}{E^2}\frac{dE}{d\eta}+2C_\nu\frac{K}{E}\frac{dK}{d\eta}+C_\nu\frac{K^2}{E}\frac{1}{\eta}\right)\frac{dK}{d\eta}+$$

$$+\left[1+C_\nu\frac{K}{E}\left(\frac{df}{d\eta}\right)^2+C_\nu\frac{K}{E}\frac{f^2}{\eta^2}+2C_\nu\frac{Kf}{\eta E}\frac{df}{d\eta}\right]K-E=0,$$

$$\left(1+C_\nu\frac{K^2}{E}\right)\frac{d^2E}{d\eta^2}+\left(\frac{\eta}{2}+\frac{1}{\eta}-C_\nu\frac{K^2}{E^2}\frac{dE}{d\eta}+2C_\nu\frac{K}{E}\frac{dK}{d\eta}+C_\nu\frac{K^2}{E}\frac{1}{\eta}\right)\frac{dE}{d\eta}+\left(2-C_{2\varepsilon}\frac{E}{K}\right)E+$$

$$+C_{1\varepsilon}C_\nu K\left[\left(\frac{df}{d\eta}\right)^2+\left(\frac{f}{\eta}\right)^2+2\frac{df}{d\eta}\frac{f}{\eta}\right]=0\,.$$

Provided that Eq. (18) is satisfied, the equation of motion takes the form

$$\left(1+C_\nu\frac{K^2}{E}\right)\frac{d^2f}{d\eta^2}+\left(\frac{\eta}{2}+\frac{1}{\eta}-C_\nu\frac{K^2}{E^2}\frac{dE}{d\eta}+2C_\nu\frac{K}{E}\frac{dK}{d\eta}+C_\nu\frac{K^2}{E}\frac{1}{\eta}\right)\frac{df}{d\eta}+$$

$$+\left(\frac{1}{2}-\frac{1}{\eta^2}-C_\nu\frac{K^2}{E}\frac{1}{\eta^2}+C_\nu\frac{K^2}{\eta E^2}\frac{dE}{d\eta}-2C_\nu\frac{K}{\eta E}\frac{dK}{d\eta}\right)f=-\frac{Gt^{3/2}}{\sqrt{\nu}}\,.$$

Now we consider two-dimensional flows. First, take the channel with a rectangle cross section with a laminar mode of flow. The equation of motion has the form

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\left(\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}\right)=G_0t^n+\nu\left(\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}\right).$$

Its symmetries are described by the expression

$$q = (2C_1 (n+1)\, u + C_2)\, \partial_u + C_1 2t\partial_t + (C_1 y + C_3)\, \partial_y + (C_1 y + C_4)\, \partial_z\,.$$

Consequently, the self-similar forms have the form

$$\eta = \frac{y}{\sqrt{\nu t}}\,,\quad \xi = \frac{z}{\sqrt{\nu t}}\,,\quad u = \frac{\nu}{h} f(\eta, \xi)\, \frac{t^{n+1}\nu^{n+1}}{h^{2n+2}}\,.$$

They allow us to obtain the equation

$$\frac{\partial^2 f}{\partial \eta^2} + \frac{\partial^2 f}{\partial \xi^2} + \frac{\eta}{2}\frac{\partial f}{\partial \eta} + \frac{\xi}{2}\frac{\partial f}{\partial \xi} - f(n+1) = -A\,,$$

which should be integrated under the boundary conditions

$$f = 0 \quad \text{for} \quad y = 0\,,\ \ y = h\,,\ \ f = 0 \quad \text{for} \quad z = 0\,,\ \ z = b\,,$$

where h is the width of the channel in the y direction and b in the z direction, and the time is used as the parameter.

For the curvilinear two-dimensional channel the following relations are valid:

• equation of motion

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho r}\frac{\partial p}{\partial \varphi} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right) = \frac{G_0 t^n}{r} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right),$$

• infinitesimal operator

$$q = C_1 (2n+1)\, u\partial_u + C_1 2t\partial_t + C_1 r\partial_r + (C_1 z + C_2)\, \partial_z\,,$$

• self-similar variables

$$\eta = \frac{r}{\sqrt{\nu t}}\,,\quad \xi = \frac{z}{\sqrt{\nu t}}\,,\quad u = \frac{\nu}{R_1} f(\eta, \xi)\, \frac{t^{\frac{2n+1}{2}}\nu^{\frac{2n+1}{2}}}{R_1^{2n+1}}\,,$$

• self-similar equation

$$\frac{\partial^2 f}{\partial \eta^2} + \frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\eta}{2} + \frac{1}{\eta}\right)\frac{\partial f}{\partial \eta} + \frac{\xi}{2}\frac{\partial f}{\partial \xi} - f\left(n + \frac{1}{2} + \frac{1}{\eta^2}\right) = -\frac{B}{\eta}\,,$$

• boundary conditions

$$f = 0 \quad \text{for} \quad r = R_1\,,\ \ r = R_2\,,\ \ f = 0 \quad \text{for} \quad z = 0\,,\ \ z = b\,.$$

The two-dimensional turbulent flow will be analyzed with the aid of the k–ε model, since the other two models that were used in the present investigation for one-dimensional flows are inconve-nient in this case. For a rectangular channel flow the model has the form

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{\partial}{\partial y}\left[\left(\nu + C_\nu \frac{k^2}{\varepsilon}\right)\frac{\partial u}{\partial y}\right] + \frac{\partial}{\partial z}\left[\left(\nu + C_\nu \frac{k^2}{\varepsilon}\right)\frac{\partial u}{\partial z}\right] =$$

$$= G_0 t^n + \frac{\partial}{\partial y}\left[\left(\nu + C_\nu \frac{k^2}{\varepsilon}\right)\frac{\partial u}{\partial y}\right] + \frac{\partial}{\partial z}\left[\left(\nu + C_\nu \frac{k^2}{\varepsilon}\right)\frac{\partial u}{\partial z}\right],$$

$$\frac{\partial k}{\partial t} = \frac{\partial}{\partial y}\left[\left(\nu + C_\nu \frac{k^2}{\varepsilon}\right)\frac{\partial k}{\partial y}\right] + \frac{\partial}{\partial z}\left[\left(\nu + C_\nu \frac{k^2}{\varepsilon}\right)\frac{\partial k}{\partial z}\right] + C_\nu \frac{k^2}{\varepsilon}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] - \varepsilon\,,$$

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial}{\partial y}\left[\left(\nu + C_\nu \frac{k^2}{\varepsilon}\right)\frac{\partial \varepsilon}{\partial y}\right] + \frac{\partial}{\partial z}\left[\left(\nu + C_\nu \frac{k^2}{\varepsilon}\right)\frac{\partial \varepsilon}{\partial z}\right] + C_\nu C_{1\varepsilon} k\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] - C_{2\varepsilon}\frac{\varepsilon^2}{k}\,.$$

This system of equations for $n = -1.5$ has the symmetries

$$q = \left(\frac{1}{2} C_1 y + C_2\right)\partial_y + \left(\frac{1}{2} C_1 z + C_3\right)\partial_z + C_1 t\partial_t - \left(\frac{1}{2} C_1 u + C_4\right)\partial_u - C_1 k\partial_k - 2C_1\varepsilon\partial_\varepsilon\,,$$

which generate the following self-similar forms:

$$\eta = \frac{y}{\sqrt{\nu t}},\quad \xi = \frac{z}{\sqrt{\nu t}},\quad u = \sqrt{\frac{\nu}{t}}\, f(\eta, \xi)\,,\quad k = \frac{\nu}{t} K(\eta, \xi)\,,\quad \varepsilon = \frac{\nu}{t^2} E(\eta, \xi)\,,$$

$$\left(1 + C_\nu \frac{K^2}{E}\right)\frac{\partial^2 f}{\partial \eta^2} + \left(1 + C_\nu \frac{K^2}{E}\right)\frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\eta}{2} - C_\nu \frac{K^2}{E^2}\frac{\partial E}{\partial \eta} + 2C_\nu \frac{K}{E}\frac{\partial K}{\partial \eta}\right)\frac{\partial f}{\partial \eta} +$$

$$+\left(\frac{\xi}{2} - C_\nu \frac{K^2}{E^2}\frac{\partial E}{\partial \xi} + 2C_\nu \frac{K}{E}\frac{\partial K}{\partial \xi}\right)\frac{\partial f}{\partial \xi} + \frac{1}{2} f = -\frac{G_0 t^{3/2+n}}{\sqrt{\nu}}\,,$$

$$\left(1 + C_\nu \frac{K^2}{E}\right)\frac{\partial^2 K}{\partial \eta^2} + \left(1 + C_\nu \frac{K^2}{E}\right)\frac{\partial^2 K}{\partial \xi^2} + \left(\frac{\eta}{2} - C_\nu \frac{K^2}{E^2}\frac{\partial E}{\partial \eta} + 2C_\nu \frac{K}{E}\frac{\partial K}{\partial \eta}\right)\frac{\partial K}{\partial \eta} +$$

$$+\left(\frac{\eta}{2} - C_\nu \frac{K^2}{E^2}\frac{\partial E}{\partial \xi} + 2C_\nu \frac{K}{E}\frac{\partial K}{\partial \xi}\right)\frac{\partial K}{\partial \xi} + \left[1 + C_\nu \frac{K}{E}\left(\frac{\partial f}{\partial \eta}\right)^2 + C_\nu \frac{K}{E}\left(\frac{\partial f}{\partial \xi}\right)^2\right]K - E = 0\,,$$

$$\left(1 + C_\nu \frac{K^2}{E}\right)\frac{\partial^2 E}{\partial \eta^2} + \left(1 + C_\nu \frac{K^2}{E}\right)\frac{\partial^2 E}{\partial \xi^2} + \left(\frac{\eta}{2} - C_\nu \frac{K^2}{E^2}\frac{\partial E}{\partial \eta} + 2C_\nu \frac{K}{E}\frac{\partial K}{\partial \eta}\right)\frac{\partial E}{\partial \eta} +$$

$$+\left(\frac{\eta}{2} - C_\nu \frac{K^2}{E^2}\frac{\partial E}{\partial \xi} + 2C_\nu \frac{K}{E}\frac{\partial K}{\partial \xi}\right)\frac{\partial E}{\partial \xi} + \left(2 - C_{2\varepsilon}\frac{E}{K}\right)E + C_{1\varepsilon} C_\nu K\left[\left(\frac{\partial f}{\partial \eta}\right)^2 + \left(\frac{\partial f}{\partial \xi}\right)^2\right] = 0$$

When Eq. (18) is satisfied, for the equation of motion we have

$$\left(1 + C_\nu \frac{K^2}{E}\right)\frac{\partial^2 f}{\partial \eta^2} + \left(1 + C_\nu \frac{K^2}{E}\right)\frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\eta}{2} - C_\nu \frac{K^2}{E^2}\frac{\partial E}{\partial \eta} + 2C_\nu \frac{K}{E}\frac{\partial K}{\partial \eta}\right)\frac{\partial f}{\partial \eta} +$$

$$+\left(\frac{\xi}{2} - C_\nu \frac{K^2}{E^2}\frac{\partial E}{\partial \xi} + 2C_\nu \frac{K}{E}\frac{\partial K}{\partial \xi}\right)\frac{\partial f}{\partial \xi} + \frac{1}{2} f = -\frac{G t^{3/2}}{\sqrt{\nu}}\,.$$

The last of the cases considered is a turbulent two-dimensional curvilinear channel flow. The k–ε model here has the form

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{dp}{dx}+\frac{1}{r}\frac{\partial}{\partial y}\left[\left(\nu+C_\nu\frac{k^2}{\varepsilon}\right)\left(r\frac{\partial u}{\partial r}-u\right)\right]+\left(\nu+C_\nu\frac{k^2}{\varepsilon}\right)\frac{\partial}{\partial r}\left[\frac{u}{r}\right]+\frac{\partial}{\partial z}\left[\left(\nu+C_\nu\frac{k^2}{\varepsilon}\right)\frac{\partial u}{\partial z}\right]=$$

$$=G_0t^n+\frac{1}{r}\frac{\partial}{\partial r}\left[\left(\nu+C_\nu\frac{k^2}{\varepsilon}\right)\left(r\frac{\partial u}{\partial r}-u\right)\right]+\left(\nu+C_\nu\frac{k^2}{\varepsilon}\right)\frac{\partial}{\partial r}\left[\frac{u}{r}\right]+\frac{\partial}{\partial z}\left[\left(\nu+C_\nu\frac{k^2}{\varepsilon}\right)\frac{\partial u}{\partial z}\right],$$

$$\frac{\partial k}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left[\left(\nu+C_\nu\frac{k^2}{\varepsilon}\right)r\frac{\partial k}{\partial r}\right]+\frac{\partial}{\partial z}\left[\left(\nu+C_\nu\frac{k^2}{\varepsilon}\right)\frac{\partial k}{\partial z}\right]+C_\nu\frac{k^2}{\varepsilon}\left[\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right]-\varepsilon\,,$$

$$\frac{\partial \varepsilon}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left[\left(\nu+C_\nu\frac{k^2}{\varepsilon}\right)r\frac{\partial \varepsilon}{\partial r}\right]+\frac{\partial}{\partial z}\left[\left(\nu+C_\nu\frac{k^2}{\varepsilon}\right)\frac{\partial \varepsilon}{\partial z}\right]+C_\nu C_{1\varepsilon}k\left[\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right]-C_{2\varepsilon}\frac{\varepsilon^2}{k}\,.$$

The symmetries of this system for $n = -1.5$ are characterized by the expression

$$q=\frac{1}{2}r\partial_r+\frac{1}{2}z\partial_z+t\partial_t-\frac{1}{2}u\partial_u-k\partial_k-2\varepsilon\partial_\varepsilon\,.$$

The self-similar forms are accordingly the following:

$$\eta=\frac{r}{\sqrt{\nu t}}\,,\quad \xi=\frac{z}{\sqrt{\nu t}}\,,\quad u=\sqrt{\frac{\nu}{t}}\,f\,(\eta,\xi)\,,\quad k=\frac{\nu}{t}K\,(\eta,\xi)\,,\quad \varepsilon=\frac{\nu}{t^2}E\,(\eta,\xi)\,,$$

$$\left(1+C_\nu\frac{K^2}{E}\right)\frac{\partial^2 f}{\partial\eta^2}+\left(1+C_\nu\frac{K^2}{E}\right)\frac{\partial^2 f}{\partial\xi^2}+\left(\frac{\eta}{2}+\frac{1}{\eta}-C_\nu\frac{K^2}{E^2}\frac{\partial E}{\partial\eta}+2C_\nu\frac{K}{E}\frac{\partial K}{\partial\eta}+C_\nu\frac{K^2}{E}\frac{1}{\eta}\right)\frac{\partial f}{\partial\eta}+$$

$$+\left(\frac{\xi}{2}-C_\nu\frac{K^2}{E^2}\frac{\partial E}{\partial\xi}+2C_\nu\frac{K}{E}\frac{\partial K}{\partial\xi}\right)\frac{\partial f}{\partial\xi}+\left(\frac{1}{2}-\frac{1}{\eta^2}-C_\nu\frac{K^2}{E}\frac{1}{\eta^2}+C_\nu\frac{K^2}{\eta E^2}\frac{\partial E}{\partial\eta}-2C_\nu\frac{K}{\eta E}\frac{\partial K}{\partial\eta}\right)f=-\frac{G_0t^{3/2+n}}{\sqrt{\nu}}\,,$$

$$\left(1+C_\nu\frac{K^2}{E}\right)\frac{\partial^2 K}{\partial\eta^2}+\left(1+C_\nu\frac{K^2}{E}\right)\frac{\partial^2 K}{\partial\xi^2}+\left(\frac{\eta}{2}+\frac{1}{\eta}-C_\nu\frac{K^2}{E^2}\frac{\partial E}{\partial\eta}+2C_\nu\frac{K}{E}\frac{\partial K}{\partial\eta}+C_\nu\frac{K^2}{E}\frac{1}{\eta}\right)\frac{\partial K}{\partial\eta}+$$

$$+\left(\frac{\xi}{2}-C_\nu\frac{K^2}{E^2}\frac{\partial E}{\partial\xi}+2C_\nu\frac{K}{E}\frac{\partial K}{\partial\xi}\right)\frac{\partial K}{\partial\xi}+\left(1+C_\nu\frac{K}{E}\left[\left(\frac{\partial f}{\partial\eta}\right)^2+\left(\frac{\partial f}{\partial\xi}\right)^2\right]+C_\nu\frac{K}{E}\frac{f^2}{\eta^2}+2C_\nu\frac{Kf}{\eta E}\frac{\partial f}{\partial\eta}\right]K-E=0\,,$$

$$\left(1+C_\nu\frac{K^2}{E}\right)\frac{\partial^2 E}{\partial\eta^2}+\left(1+C_\nu\frac{K^2}{E}\right)\frac{\partial^2 E}{\partial\xi^2}+\left(\frac{\eta}{2}+\frac{1}{\eta}-C_\nu\frac{K^2}{E^2}\frac{\partial E}{\partial\eta}+2C_\nu\frac{K}{E}\frac{\partial K}{\partial\eta}+C_\nu\frac{K^2}{E}\frac{1}{\eta}\right)\frac{\partial E}{\partial\eta}+$$

$$+\left(\frac{\xi}{2}-C_\nu\frac{K^2}{E^2}\frac{\partial E}{\partial\xi}+2C_\nu\frac{K}{E}\frac{\partial K}{\partial\xi}\right)\frac{\partial E}{\partial\xi}+\left(2-C_{2\varepsilon}\frac{E}{K}\right)E+C_{1\varepsilon}C_\nu K\left[\left(\frac{\partial f}{\partial\eta}\right)^2+\left(\frac{f}{\eta}\right)^2+\left(\frac{\partial f}{\partial\xi}\right)^2+2\frac{\partial f}{\partial\eta}\frac{f}{\eta}\right]=0\,.$$

Once Eq. (18) is satisfied, the equation of motion takes the form

TABLE 1. Self-Similar Forms of Nonstationary Channel Flows

Regime	Channel	η	ξ	u	ν_Σ	k	ε
Laminar	Plane, one-dimensional	$y/\sqrt{\nu t}$	–	$\frac{\nu}{h} f(\eta) \frac{t^{n+1}\nu^{n+1}}{h^{2n+2}}$	–	–	–
Laminar	Cylindrical, one-dimensional	$r/\sqrt{\nu t}$	–	$\frac{\nu}{R} f(\eta) \frac{t^{n+1}\nu^{n+1}}{R^{2n+2}}$	–	–	–
Laminar	Curvilinear, one-dimensional	$r/\sqrt{\nu t}$	–	$\frac{\nu}{R_1} f(\eta) \frac{t^{(2n+1)/2}\nu^{(2n+1)/2}}{R_1^{2n+2}}$	–	–	–
Turbulent, mixing-path length	Plane, one-dimensional	$y/\sqrt{\nu t}$	–	$\sqrt{\nu/t}\, f(\eta)$	–	–	–
Turbulent, ν model	Same	$\frac{yh^{3+2n}}{(\nu t)^{2+n}}$	–	$\frac{\nu^{2+n}t^{1+n}}{h^{3+2n}} f(\eta)$	$\frac{\nu^{4+2n}t^{3+2n}}{h^{6+4n}} N(\eta)$	–	–
Turbulent, k–ε model	»	$y/\sqrt{\nu t}$	–	$\sqrt{\nu/t}\, f(\eta)$	–	$(\nu/t)K(\eta)$	$(\nu/t^2)E(\eta)$
Turbulent, k–ε model	Cylindrical, one-dimensional	$r/\sqrt{\nu t}$	–	$\sqrt{\nu/t}\, f(\eta)$	–	$(\nu/t)K(\eta)$	$(\nu/t^2)E(\eta)$
Turbulent, k–ε model	Curvilinear, one-dimensional	$r/\sqrt{\nu t}$	–	$\sqrt{\nu/t}\, f(\eta)$	–	$(\nu/t)K(\eta)$	$(\nu/t^2)E(\eta)$
Laminar	Rectangular, two-dimensional	$y/\sqrt{\nu t}$	$z/\sqrt{\nu t}$	$\frac{\nu}{h} f(\eta, \xi) \frac{t^{n+1}\nu^{n+1}}{h^{2n+2}}$	–	–	–
Laminar	Curvilinear, two-dimensional	$r/\sqrt{\nu t}$	$z/\sqrt{\nu t}$	$\frac{\nu}{R_1} f(\eta, \xi) \frac{t^{(2n+1)/2}\nu^{(2n+1)/2}}{R_1^{2n+2}}$	–	–	–
Turbulent, k–ε model	Rectangular, two-dimensional	$y/\sqrt{\nu t}$	$z/\sqrt{\nu t}$	$\sqrt{\nu/t}\, f(\eta)$	–	$(\nu/t)K(\eta)$	$(\nu/t^2)E(\eta)$
Turbulent, k–ε model	Curvilinear, two-dimensional	$r/\sqrt{\nu t}$	$z/\sqrt{\nu t}$	$\sqrt{\nu/t}\, f(\eta)$	–	$(\nu/t)K(\eta)$	$(\nu/t^2)E(\eta)$

$$\left(1 + C_\nu \frac{K^2}{E}\right)\frac{\partial^2 f}{\partial \eta^2} + \left(1 + C_\nu \frac{K^2}{E}\right)\frac{\partial^2 f}{\partial \xi^2} + \left(\frac{\eta}{2} + \frac{1}{\eta} - C_\nu \frac{K^2}{E^2}\frac{\partial E}{\partial \eta} + 2C_\nu \frac{K}{E}\frac{\partial K}{\partial \eta} + C_\nu \frac{K^2}{E}\frac{1}{\eta}\right)\frac{\partial f}{\partial \eta} +$$

$$+ \left(\frac{\xi}{2} - C_\nu \frac{K^2}{E^2}\frac{\partial E}{\partial \xi} + 2C_\nu \frac{K}{E}\frac{\partial K}{\partial \xi}\right)\frac{\partial f}{\partial \xi} + \left(\frac{1}{2} - \frac{1}{\eta^2} - C_\nu \frac{K^2}{E}\frac{1}{\eta^2} + C_\nu \frac{K^2}{\eta E^2}\frac{\partial E}{\partial \eta} - 2C_\nu \frac{K}{\eta E}\frac{\partial K}{\partial \eta}\right) f = -\frac{G t^{3/2}}{\sqrt{\nu}} .$$

The results of the analysis of all the cases considered are given in Table 1.

The present investigation has shown that the use of symmetries makes it possible to rather easily analyze a wide range of nonstationary flows in differently shaped channels. One-dimensional nonstationary problems are then reduced to ordinary differential equations, and two-dimensional ones, to equations in partial derivatives with two self-similar variables. If in these equations the time is converted as $t = x/U$, where U is the velocity at the channel inlet, then the relations obtained can be used to analyze flows over the starting length of the channel (as in [1]), provided that the pressure gradient is described by the Heaviside function. It should be noted that this kind of calculation method based on the symmetries of the process can be easily extended to other kinds of nonstationary flows, as, e.g., Couette flows, free convective flows, etc.

NOTATION

b, width of a channel in the *z* direction; *h*, width of a channel in the *y* direction; *k*, kinetic energy of turbulence; *p*, pressure; *r*, φ, *z*, cylindrical coordinates; R_1 and R_2, external and internal radii for the annular channel, the radii of the convex and concave walls for the curvilinear channel; *t*, time; *u*, longitudinal velocity component; *x*, coordinate along the flow; *y*, *z*, transverse coordinates; ε, dissipation rate; ν, kinematic viscosity; ν_Σ, total viscosity; ρ, density.

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